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SPACE SCIENCES LABORATORY

October 1960

ORBIT DETERMINATION OF A NON-TRANSMITTING SATELLITE  
USING DOPPLER TRACKING DATA

(Final Report)

By

Dr. Paul B. Richards

Shepherd Program, ARPA Order 8-58, BRL DOPLOC Satellite Fence  
Series, Report No. 12 in the Series.

Work performed under contract No. DA-36-034-509-ORD-3159-RD, for  
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ABSTRACT

A method is presented for predicting the orbit of a passive (non-transmitting) satellite, using exclusively Doppler data collected during one passage of the satellite by a tracking system consisting of only one transmitter and one receiver. Equations are derived which relate passive satellite Doppler tracking data to range, speed and time parameters at closest approach. These determine an ellipsoid whose foci are the transmitter and receiver stations. The satellite lies on this ellipsoid, and its trajectory is tangent to the surface. The satellite position and velocity components at closest approach are calculated on a digital computer using information contained in the Doppler data together with the geometry of the tracking system. Preliminary orbits determined in this way agree favorably with the refined orbits of BRL and Space Track.

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## ORBIT DETERMINATION OF A NON-TRANSMITTING SATELLITE USING DOPPLER TRACKING DATA

### 1. INTRODUCTION

In connection with the application of radio-Doppler methods to satellite tracking, the Ballistic Research Laboratories, Aberdeen Proving Ground, requested the Space Sciences Laboratory of General Electric to investigate the problem of orbit determination of a passive (non-transmitting) earth satellite using exclusively Doppler data collected during one pass of the vehicle.\*

The basic problem in orbit determination is to determine the components, with respect to a geocentric inertial coordinate system, of position and velocity at a particular time for the orbit being observed, i. e., determine  $x(t_0)$ ,  $y(t_0)$ ,  $z(t_0)$ ,  $\dot{x}(t_0)$ ,  $\dot{y}(t_0)$ ,  $\dot{z}(t_0)$ , for a given time  $t_0$ . For an earth satellite, this is equivalent to determining the orbital elements  $a$ ,  $e$ ,  $i$ ,  $\Omega$ ,  $\omega$ ,  $M(t_0)$ , where

- $a$  is the semi-major axis,
- $e$  is the eccentricity,
- $i$  is the inclination of the orbital plane,
- $\Omega$  is the right ascension of the ascending node,
- $\omega$  is the argument of perigee,
- $M(t_0)$  is the mean anomaly at  $t_0$ .

The equation which describes the Doppler shift of radio signals transmitted from a station on the earth, reflected from the passive

\*Contract No. DA-36-034-509-ORD-3159-RD.

satellite, and received at another station on the earth is\*

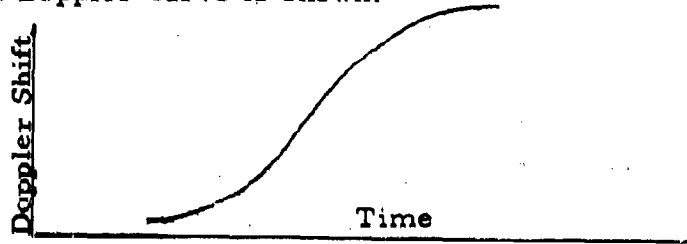
$$\dot{\rho}_T + \dot{\rho}_R = \frac{c}{f_T} (f_T - f_R),$$

where  $\dot{\rho}_T$  and  $\dot{\rho}_R$  are the range rates of the satellite relative to the transmitter and receiver stations, respectively,

$f_T$  and  $f_R$  are the transmitter and receiver frequencies, respectively,

and  $c$  is the speed of light.

A typical Doppler curve is shown.



The measured Doppler data  $(f_T - f_R)$  is related to the solution  $x(t_0)$ ,  $y(t_0)$ ,  $z(t_0)$ ,  $\dot{x}(t_0)$ ,  $\dot{y}(t_0)$ ,  $\dot{z}(t_0)$  of the orbit determination problem by the expressions relating the time-dependent range rates to the time-dependent solution  $x(t)$ ,  $y(t)$ ,  $z(t)$ ,  $\dot{x}(t)$ ,  $\dot{y}(t)$ ,  $\dot{z}(t)$  of the equation of motion of the satellite,  $\ddot{\mathbf{r}} + \mu \mathbf{r}/r^3 = 0$ , since the solution of the equation of motion is unique, corresponding to a particular set of initial conditions  $x(t_0)$ ,  $y(t_0)$ ,  $z(t_0)$ ,  $\dot{x}(t_0)$ ,  $\dot{y}(t_0)$ ,  $\dot{z}(t_0)$ . The desired expression for  $\dot{\rho}_T$  is

$$\dot{\rho}_T(t) = \frac{(x(t) - X_T(t)) (\dot{x}(t) - \dot{X}_T(t)) + (y(t) - Y_T(t)) (\dot{y}(t) - \dot{Y}_T(t)) + (z(t) - Z_T(t)) (\dot{z}(t) - \dot{Z}_T(t))}{\left[ (x(t) - X_T(t))^2 + (y(t) - Y_T(t))^2 + (z(t) - Z_T(t))^2 \right]^{1/2}}$$

where  $X_T$ ,  $Y_T$ ,  $Z_T$  are the position components of the transmitter,

$\dot{X}_T$ ,  $\dot{Y}_T$ ,  $\dot{Z}_T$  are the velocity components of the transmitter.

Similarly for  $\dot{\rho}_R(t)$ .

\*For derivation of the Doppler equation see Appendix.

Having obtained Doppler data from one or more transmitter-receiver stations, the procedure for arriving at a solution to the orbit determination problem is as follows:

1. Select preliminary values of the initial conditions  $x(t_0)$ ,  $y(t_0)$ ,  $z(t_0)$ ,  $\dot{x}(t_0)$ ,  $\dot{y}(t_0)$ ,  $\dot{z}(t_0)$ , i. e., establish a preliminary orbit.
2. Solve the equation of motion for the time-dependent position and velocity components.
3. Compute values of  $\dot{\rho}_T + \dot{\rho}_R$  for each transmitter-receiver pair at times corresponding to measured Doppler data.
4. Compare computed values of  $\dot{\rho}_T + \dot{\rho}_R$  with data.
5. Select an improved set of initial conditions and repeat the process until the computed values agree with the data.

Items 2 to 5 constitute the determination of the refined or final orbit.

The Ballistic Research Laboratories has a numerical technique for establishing the final orbit that will converge with data from a single transmitter-receiver system provided the preliminary values of the initial conditions are sufficiently accurate [ 1 ].\*

Doppler tracking of active (transmitting) satellites has received considerable attention [ 2, 3, 4, 5, 6 ], but little has been found in the literature regarding the problem of establishing preliminary orbits.

For these reasons, emphasis at the Space Sciences Laboratory has been placed on obtaining a reliable method for determining a preliminary orbit of a passive satellite, using only Doppler data collected from a single transmitter-receiver system during one passage of the satellite. The method is formulated in this report, and applied to actual Doppler data.

\*Numbers in brackets refer to bibliography at end of paper.



The geometry of such a system can cause an ambiguity in the position and velocity components of the satellite at a given time. For a specific altitude, the satellite can be at one of four possible positions. These positions consist of two pairs of points, each pair consisting of points symmetric with respect to the midpoint of the base line between the transmitter and receiver. For each such pair, the velocity components are uniquely determined.

It is shown, however, that the transmitter antenna beam can be oriented so that ambiguity in the direction of motion of the satellite (signs of velocity components) can be completely eliminated. Furthermore, severe restrictions can be imposed on the position coordinates by the speed of the satellite and the location of the satellite relative to the antenna beams at the given time.

Thus, ambiguities caused by the basic geometry can be considerably, if not completely, eliminated by judicious orientation of the antenna beams and by using all possible information contained in the data.

The theory is formulated analytically, and computation can be performed on a digital computer. This suggests the possibility that the orbit of a passive satellite can be determined using only Doppler data from a single transmitter-receiver system while the satellite is still visible above the horizon.

The application of this theory to active satellites is also described.

## 2. ESTABLISHING A PRELIMINARY ORBIT FOR PASSIVE SATELLITES

Integration of the Doppler equation for a single transmitter-receiver pair,

$$\dot{\rho}_T + \dot{\rho}_R = \lambda_T \Delta f, \quad \text{where } \lambda_T = \frac{c}{f_T}, \quad \Delta f = f_T - f_R, \quad (2.1)$$

yields

$$\rho_T + \rho_R = \int_{t_0}^t \lambda_T \Delta f dt + (\rho_T + \rho_R)_{t_0} = 2a(t), \quad (2.2)$$

which indicates that, at a given time  $t$ , the satellite lies on an ellipsoid of semi-major axis  $a(t)$ , with foci at  $T$  and  $R$ , the sites of the transmitter and receiver. Since  $T$  and  $R$  are known locations, the eccentricity of the ellipsoid is obtained from the expression  $ae = \overline{TR}/2$ . Thus the ellipsoid is completely determined once the constant of integration  $(\rho_T + \rho_R)_{t_0}$  is known. This is determined in the following manner: The Doppler equation (2.1) may be written in vector form as

$$\frac{\bar{\rho}_T \cdot \dot{\bar{\rho}}_T}{\rho_T} + \frac{\bar{\rho}_R \cdot \dot{\bar{\rho}}_R}{\rho_R} = \lambda_T \Delta f. \quad (2.3)$$

In a geocentric inertial coordinate system, neglecting the earth's rotation,\*  $\dot{\bar{\rho}}_T = \dot{\bar{\rho}}_R = \bar{v}$ , where  $\bar{v}$  is the satellite velocity, and equation (2.3) becomes

$$\bar{v} \cdot \left( \frac{\bar{\rho}_T}{\rho_T} + \frac{\bar{\rho}_R}{\rho_R} \right) = \lambda_T \Delta f. \quad (2.4)$$

Define the vector

$$\bar{p}(t) = \frac{1}{2} \left[ \frac{\rho_T \rho_R}{1 - \left( \frac{\overline{TR}}{\rho_T + \rho_R} \right)^2} \right]^{1/2} \left( \frac{\bar{\rho}_T}{\rho_T} + \frac{\bar{\rho}_R}{\rho_R} \right). \quad (2.5)$$

Then

$$|\bar{p}(t)| = p(t) = \frac{1}{2} (\rho_T + \rho_R), \quad (2.6)$$

\*For a low earth satellite, the error incurred by this assumption will be small.

and the Doppler equation (2. 1) becomes

$$\dot{p}(t) = \frac{\lambda_T \Delta f}{2} \quad (2. 7)$$

or

$$\frac{\bar{p} \cdot \dot{\bar{p}}}{p} = \frac{\lambda_T \Delta f}{2} \quad (2. 8)$$

Define the time of closest approach as the time at which  $\rho_T + \rho_R$  is minimum (  $\Delta f = 0$  ) and let  $t_0$  in (2. 2) be this time. Then the constant of integration in (2. 2) is the major axis of the smallest ellipsoid upon which the satellite will lie, the ellipsoid at closest approach. From (2. 2) and (2. 6),

$$p_0 = p(t_0) = \frac{1}{2} (\rho_T + \rho_R)_{t_0} = a(t_0).$$

Using zero subscripts to denote values at  $t_0$ , the time of closest approach, it follows from (2. 4), (2. 5) and (2. 8) that

$$\bar{v}_0 \perp \bar{p}_0 \quad (2. 9)$$

$$\dot{\bar{p}}_0 \perp \bar{p}_0. \quad (2. 10)$$

Further, the vector  $\bar{p}(t)$  is perpendicular to the tangent plane of the ellipsoid at the location of the satellite. Hence, both vectors  $\bar{v}_0$  and  $\dot{\bar{p}}_0$  lie in the plane which is tangent to the ellipsoid at closest approach, at the location of the satellite.

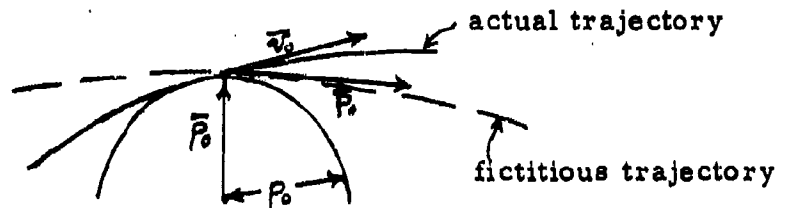
Thus, the following conclusions have been established:

At the time of closest approach  $t_0$  the satellite lies on the surface of the ellipsoid of semi-major axis  $p_0$ , having foci at T and R, the transmitter and receiver; further, the satellite trajectory is tangent to this ellipsoid. The location of the satellite on this ellipsoid, together with its associated velocity and the time  $t_0$ , will determine a preliminary orbit of the satellite.

Before proceeding to determine  $p_0$  and the satellite position and velocity components at  $t_0$ , an interpretation of  $\dot{\bar{p}}(t)$  is in order. This vector can be expressed in the form

$$\dot{\bar{p}}(t) = a \bar{e}_T + b \bar{e}_R + c \bar{v},$$

where  $a, b, c$  are time-dependent scalars, for the vectors  $\bar{e}_T, \bar{e}_R, \bar{v}$  constitute a basis in three-dimensional space. The vector  $\dot{\bar{p}}(t)$  may be interpreted as a fictitious velocity, for if we hold the base of  $\bar{p}(t)$  fixed at the origin of a three-dimensional vector space, the locus of the end point of  $\bar{p}(t)$  will be a fictitious trajectory. We take as origin of this vector space the base of  $\bar{p}_0$ , i.e.,  $\bar{p}$  at closest approach. This point will be in the plane of  $\bar{e}_T$  and  $\bar{e}_R$  on the normal to the actual trajectory at the point of the trajectory where  $\rho_T + \rho_R$  is a minimum, and at a distance along this normal equal to  $1/2$  this minimum value of  $\rho_T + \rho_R$ . In the vector space there will be a sphere of radius  $p_0 = 1/2 (\rho_T + \rho_R)_{min}$ , such that the fictitious trajectory and the true trajectory are both tangent to this sphere at the point of closest approach. Elsewhere both trajectories are exterior to the sphere.



If we now assume that  $\dot{\bar{p}}$  is constant ( $= \dot{\bar{p}}_0$ ) in the neighborhood of closest approach  $t_0$ , we can write, in this neighborhood,

$$\bar{p}(t) = \bar{p}_0 + \dot{\bar{p}}_0 (t - t_0) \quad (2.11)$$

Then, using (2.10)

$$p^2(t) = p_0^2 + |\dot{\bar{p}}_0|^2 (t - t_0)^2 \quad (2.12)$$

and

$$\dot{p}(t) = \frac{|\dot{\bar{p}}_0|^2 (t - t_0)}{[p_0^2 + |\dot{\bar{p}}_0|^2 (t - t_0)^2]^{1/2}} \quad (2.13)$$

Determination of Closest-Approach Parameters. - If the measured data consists of values of frequency shift  $\Delta f$  at specific times, equation (2.7) shows that the parameters  $t_0$ ,  $p_0$ ,  $|\dot{p}_0|$  can be computed using (2.13). If the data consists of average values of frequency shift ( $\overline{\Delta f}$ ) over specific time intervals  $\Delta t$ , the mean value theorem together with (2.7) and (2.13) gives

$$\begin{aligned}\lambda_r \overline{\Delta f} \Delta t &= \lambda_r \int_{\Delta t} \Delta f dt = 2 \int_{\Delta t} \dot{p} dt \\ &= 2 \left[ p_0^2 + |\dot{p}_0|^2 (t-t_0)^2 \right]^{\frac{1}{2}} \Big|_{\Delta t}\end{aligned}\quad (2.14)$$

and the desired parameters are computed from this expression.

Great care must be exercised in fitting the right side of (2.13) or (2.14) to the appropriate data. For example, (2.13) can be written as

$$\dot{p}(\tau) = \frac{\tau}{\sqrt{a + b\tau^2}}, \quad a = p_0^2/|\dot{p}_0|^4, \quad b = 1/|\dot{p}_0|^2, \quad \tau = t - t_0 \quad (2.15)$$

and for  $|\tau| < \sqrt{\frac{a}{b}} = \frac{p_0}{|\dot{p}_0|}$ ,  $\dot{p}(\tau)$  can be represented by the series

$$\dot{p}(\tau) = \frac{\tau}{\sqrt{a}} \left( 1 - \frac{1}{2} \frac{b}{a} \tau^2 + \frac{3}{8} \frac{b^2}{a^2} \tau^4 - \dots \right), \quad (2.16)$$

which indicates that  $b/a$  will be insignificant if the data used to determine  $a$  and  $b$  is linear in  $\tau$ . This can lead to inaccuracies in the computation for  $b$ . Similarly, (2.14) can be written

$$\frac{1}{2} (\lambda_r \overline{\Delta f} \Delta t) = \frac{1}{b} \left[ \sqrt{a + b\tau_2^2} - \sqrt{a + b\tau_1^2} \right],$$

$\tau_1 = t_1 - t_0$ ;  $\tau_2 = t_2 - t_0$ ;  $t_1, t_2$  end points of the interval  $\Delta t$ ,

and for  $|\tau_1|, |\tau_2| < \sqrt{\frac{a}{b}}$ ,

$$\frac{1}{2} (\lambda_r \overline{\Delta f} \Delta t) = \frac{1}{2\sqrt{a}} (\tau_2^2 - \tau_1^2) \left[ 1 - \frac{1}{4} \frac{b}{a} (\tau_2^2 + \tau_1^2) + \dots \right].$$

Since  $\tau_2 - \tau_1 = \Delta t$  does not change significantly, it is apparent that if the data is linear in the discrete variable  $T = \frac{\tau_1 + \tau_2}{2}$ , then again  $b/a$  will be very small.

Since the use of (2.13) as a representation of the Doppler curve, or S-curve as it is commonly called, is restricted to the neighborhood of closest approach, this equation cannot be fitted directly to the measured data if noise obscures non-linearities in the neighborhood of  $t_0$ . Further, formula (2.13) is symmetric about  $t_0$ , whereas the actual S-curve is asymmetric for large values of  $t$ . We therefore require an analytic representation of the measured data which is non-linear and almost symmetric for small  $\tau = t - t_0$ , to which the symmetric formula (2.13) can be equated in the neighborhood of  $t_0$ . The simplest desired representation is a polynomial fit to all the measured data.

If  $t_0$  is known, standard computational procedures can be applied to fit (2.13) or (2.15) to data. Write (2.15) as

$$\frac{\tau^2}{\dot{p}^2} = a + b\tau^2, \quad (2.17)$$

a linear expression in  $a$  and  $b$ , and the usual least-squares formulas apply.

Two different methods have been used successfully to determine  $t_0$  prior to fitting (2.15) or (2.17) to the analytical representation of the data. One method, easily programmed for a digital computer, is to fit the cubic in  $t$ ,

$$\dot{p} = a + bt + ct^2 + dt^3, \quad (2.18)$$

to the measured data in the neighborhood of  $t_0$ , with the time origin for  $t$  fairly close to  $t_0$  so that round-off errors will not be significant. Once the coefficients in (2.18) are determined,  $t_0$  is computed as the single root of the equation  $\dot{p} = 0$  that lies in this neighborhood.

An alternate procedure, which has been very successful, but more difficult to program for a digital computer, is based on formula (2.17), which indicates that if  $\tau^2/\dot{p}^2$  is plotted against  $\tau^2$ , the data will lie on a straight line with slope  $b$  and intercept  $a$ . The parameter  $t_0$  must be known in order to measure  $\tau$ , of course. If  $t_0$  is not known, denote  $\varepsilon = t'_0 - t_0$ ,  $t'_0$  the assumed time of closest approach, and let  $\xi$  be the time measured from  $t'_0$ , i.e.,  $\xi = \tau - t'_0$ . Then  $\tau = \xi + \varepsilon$  and (2.17) becomes

$$\frac{\xi^2}{\dot{p}^2} = a + b \xi^2 + a f, \quad (2.19)$$

$$f = \frac{1}{\left(1 + \frac{\varepsilon}{\xi}\right)^2} - 1.$$

Thus, if a  $t'_0 \neq t_0$  is selected as the time origin, the measured values  $\xi^2/\dot{p}^2$  will plot against  $\xi^2$  as a straight line upon which is superimposed the curve  $af$ ; the straight line will have slope  $b$  and intercept  $a$ , as before. Clearly  $\lim_{\xi \rightarrow 0} f = 0$ , which suggests that different values of  $t'_0$  be selected until one is reached which results in  $\xi^2/\dot{p}^2$  being linear in  $\xi^2$ . This is not easily adapted to digital solution, however. Instead, we proceed as follows.

Denote  $\frac{\xi^2}{\dot{p}^2} = y$ ,  $\xi^2 = \eta$ ,  $\mu = \frac{\varepsilon}{\xi}$ ,  $\varepsilon \neq 0$ ; then (2.19) becomes

$$y = a + b \eta + a f, \quad (2.20)$$

$$f = \left(\frac{\mu}{1+\mu}\right)^2 - 1, \quad \mu \neq -1 \text{ (see figure 1)}.$$

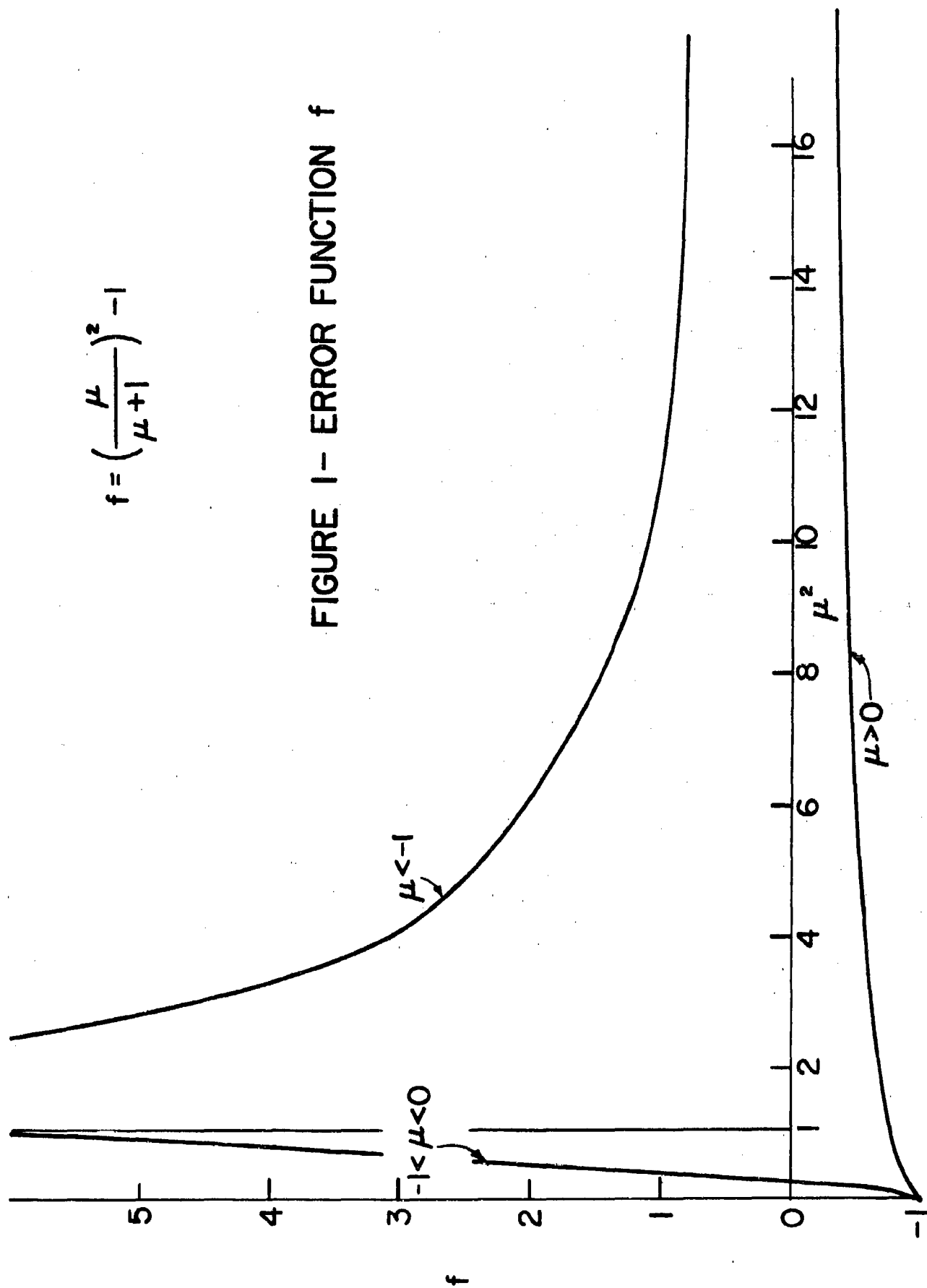
Then

$$\frac{\partial^2 y}{\partial \eta^2} = a \frac{\partial^2 f}{\partial \eta^2} = -\frac{3}{2} \frac{a}{\eta^2} \frac{\mu^3}{(1+\mu)^4}, \quad \mu \neq -1, \quad (2.21)$$

and

$$\text{sgn } \frac{\partial^2 y}{\partial \eta^2} = \text{sgn } (-\mu). \quad (2.22)$$

Now, if  $t'_0$  is chosen to the right of the correct value  $t_0$ ,  $\varepsilon = t'_0 - t_0 > 0$ .



$$f = \left( \frac{\mu}{\mu + 1} \right)^2 - 1$$

FIGURE 1- ERROR FUNCTION  $f$



Selecting values of  $\xi > 0$ ,  $\mu = \xi/\epsilon$  will be positive and, from (2.22),  $\frac{\partial^2 y}{\partial \eta^2} < 0$ . If, on the other hand,  $t_c'$  is selected to the left of the correct value  $t_0$ ,  $\epsilon < 0$ . Again using values of  $\xi > 0$ ,  $\mu$  will be negative and  $\frac{\partial^2 y}{\partial \eta^2} > 0$ . Also, it is seen from (2.21) that for  $\xi \neq 0$  and finite,  $\frac{\partial^2 y}{\partial \eta^2} = 0$  if and only if  $\epsilon = 0$ ; further,  $\frac{\partial^2 y}{\partial \eta^2}$  approaches zero monotonically from above (or below) as  $t_c'$  converges to  $t_0$  from the left (or right). Similar results hold if values of  $\xi < 0$  are chosen in both cases.

The procedure then is: (1) Choose a value of  $t_c'$ ; (2) Select three values of  $t$  to the right of  $t_c'$  and compute the second difference of  $y = \tau^2/\dot{p}^2$  at the center value of  $t$ , using measured  $\dot{p}$  data. If this second difference is negative, say, this indicates that the  $t_c'$  selected is to the right of  $t_0$ , the correct value; (3) Choose a second  $t_c'$ , to the left of the first one; (4) Using, for convenience, the same three values of  $t$  previously used, compute the second difference of  $y$  again at the center value (the values of  $\xi^2$  are now different). If this second difference is again negative, select a third  $t_c'$  farther to the left. If the second difference is positive, select the third  $t_c'$  slightly to the right of the second. In either case, repeat the procedure until the  $t_c'$  is found that makes the second difference vanish. By selecting the three values of  $t$ , used in computing the second differences, sufficiently far from the chosen  $t_c'$ , the singular situation  $\mu = -1$  is easily avoided.

This procedure suggests that  $t_0$  be determined as just discussed, and then, using (2.17),  $a$  and  $b$  be computed as the intercept and slope, respectively, of the straight line  $\tau^2/\dot{p}^2$  obtained from the measured data. Error analysis shows that the parameter  $a$  can be found in this way, but not  $b$ ; for to insure accuracy in  $p_0$  to one mile, and in  $|\dot{p}_0|$  to 100 ft/sec, it is necessary that time measurements be accurate to .4 seconds in computing the intercept  $a$ , and to .00025 seconds in computing the slope  $b$ . The latter requirement is clearly too restrictive when the

measured data consists of values of  $\dot{p}$  assigned to the midpoints of .15 second intervals.

Instead, the most successful results have been obtained by the following procedure:

- (1) Determine  $t_0$ , either by the  $f$ -function second differences (2.14) - (2.22) using measured data, or by fitting (least squares) the cubic equation (2.18) to the measured  $\dot{p}$  data in the neighborhood of  $t_0$ .

- (2) Using this  $t_0$ , use least-squares to fit the cubic

$$\dot{p} = \alpha \tau + \beta \tau^2 + \gamma \tau^3 \quad * \quad (2.23)$$

to all the measured data.

- (3) Determine  $a$ ,  $b$ , and  $p_0$  by fitting (least squares) equation (2.17) to values of  $\dot{p}$  given by (2.23) over a sufficiently large neighborhood of  $t_0$ . The interval  $t_0 \pm 30\%$  of the time interval between  $t_0$  and the side beams has been found adequate.

Determination of Position and Velocity at Closest Approach. - Once  $p_0$  is determined, the ellipsoid at the time of closest approach is known, since the eccentricity is  $e = \overline{TR}/2p_0$ , where  $\overline{TR}$  is the length of the base line between the transmitter and receiver, the foci of the ellipsoid. It remains now to determine the location of the satellite on this ellipsoid, as well as its velocity components.

We know from the previous theory that the satellite trajectory is tangent to this ellipsoid. Further information can be obtained from the Doppler data and also from the geometry of the transmitter and receiver

\*A truncated series (2.16) or a truncated Fourier series might also fit the data well.

beams. In particular, the DOPLOC\* transmitter station, located at Fort Sill, Oklahoma, consists of a 50 KW transmitter and three high-gain antennas. All of the antennas can be excited separately from the transmitter but not simultaneously. Each antenna is capable of radiating a thick fan-shaped beam having dimensions of approximately 8 x 76 degrees. The beam from the center antenna is directed vertically with the 76 degree dimension in the east-west direction. The north and south antennas are identical to the center antenna except that they are tilted to radiate fan-shaped beams 20 degrees above the north and south horizons respectively. This arrangement of beams is intended to detect a satellite as it rises over the horizon, track through the beam width, switch to the center antenna, track through its beam and then switch to the third antenna to obtain three segments of data from the characteristic S-curve.

The DOPLOC system has two receiving stations, one at the White Sands Missile Range and the other at Forrest City, Arkansas.

At each receiving station three high-gain antennas, which are identical to the transmitting antennas except in power-handling capability, are tilted to see the same volumetric space that is illuminated by the transmitting antennas. Thus each center receiving antenna has its fan-shaped beam in the east-west vertical plane but tilted toward Fort Sill. In addition to being tilted, the north and south receiving antennas at each receiving station are rotated so that each station sees approximately half of the volume of space illuminated by the north or south transmitting antenna. The receiving antennas are switched synchronously with the transmitting antennas.

In the following analysis, only one transmitter-receiver pair will be considered. Without loss of generality, we will describe the technique for the transmitter at Fort Sill, Oklahoma, and the receiver at Forrest

\*The word "DOPLOC" is a combination of the first three letters of Doppler and lock and implies a phase-locked Doppler measuring system.

City, Arkansas. These stations are separated by a base line of 435 statute miles, and of course are the foci of the ellipsoid of closest approach for each satellite observed. We will assume that this ellipsoid ( $p_0$ ) has already been determined.

A typical Doppler data plot for this transmitter-receiver pair is shown in figure 2. Here the time of closest approach occurs while the satellite is in the center beam, which means that the satellite lies on the associated ellipsoid of closest approach in the strip intersected by the center beam of the transmitter. In all cases, the ellipsoid of closest approach associated with a particular transmitter-receiver pair will be intersected by the three transmitter beams, and the location of the satellite, at time of closest approach, relative to these beams gives a good indication of where the satellite lies on the ellipsoid. Further, the time of appearance in the beams indicates the general direction of the satellite. For example, the satellite of figure 2 is clearly going south as it passes over the stations.

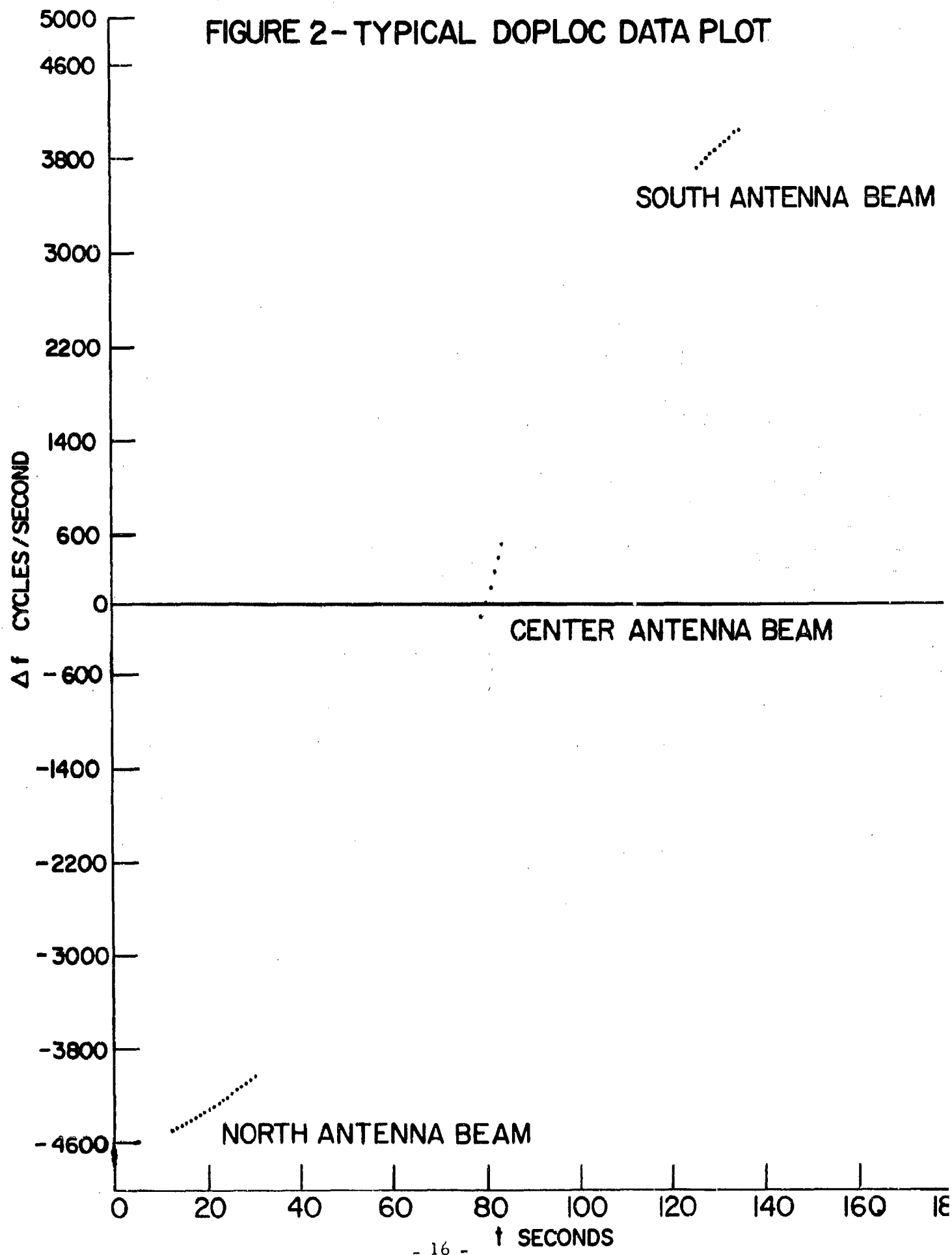
This information can be expressed analytically, in a form easily programmed for a digital computer, in order to calculate the approximate position and velocity components of the satellite at the time of closest approach.

At the time of closest approach, a horizontal plane at the altitude  $h$  of the satellite above the transmitter will intersect the transmitter beams as shown in figure 3. If we assume that the satellite remains in this plane during its flight through the three beams, and also that its speed  $v$  is that corresponding to a circular orbit at this altitude  $h$ , then the satellite path length  $\Delta s$  between the center and north beams is given by

$$\Delta s = v \Delta t \quad (2.24)$$

where  $\Delta t$  is the time interval corresponding to the gap in the data between the center and north beams. The acute angle  $\theta$  between the

FIGURE 2-TYPICAL DOPLOC DATA PLOT



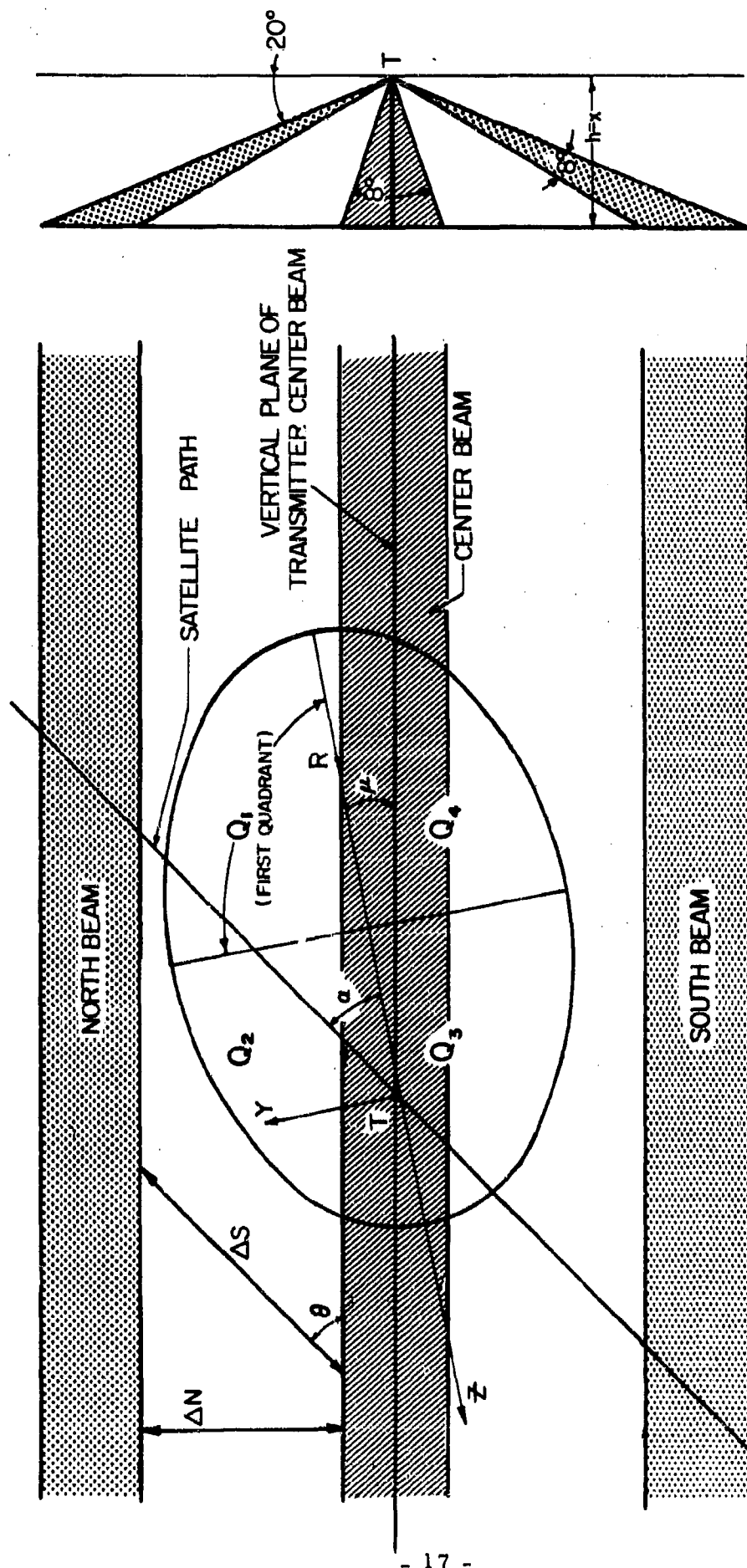


FIG.3- ORIENTATION OF ELLIPSOID AND ANTENNA BEAMS

path and the vertical plane of the transmitter center beam (in this case this plane is in the east-west direction) is obtained from

$$\Delta S = \frac{\Delta N}{\sin \theta}, \quad 0 < \theta \leq \frac{\pi}{2} \quad (2.25)$$

where  $\Delta N$  is the perpendicular distance between these beams at the given altitude  $h$ .

It will be convenient to use a topocentric coordinate system, having origin at  $T$ ,  $x$ -axis vertical,  $z$ -axis in the horizontal plane at the elevation of the transmitter and in the direction of the ellipsoid base line  $TR$  extended westward;  $y$ -axis in the same horizontal plane and normal to  $TR$  (figure 3).

Then, corresponding to a given altitude  $h$  (or  $x$ ), equations (2.24) and (2.25) give

$$\sin \theta = \frac{A x \sqrt{r+x}}{\Delta t \sqrt{k^2 M}} \quad (2.26)$$

where

- $A = \cot 28^\circ - \tan 4^\circ$
- $r = \text{radius of earth at transmitter}$
- $k^2 = \text{Newton's gravitational constant}$
- $M = \text{mass of earth.}$

Clearly the altitude  $x$  of the satellite will be restricted by the upper bound of  $\sin \theta$ . This restriction can be seen intuitively: the higher the satellite, the greater must be its speed to traverse the wider distance between beams in the fixed time interval; on the other hand, the speed in a circular orbit reduces with increased altitude. There will thus be an upper bound on permissible altitudes.

Now denote by  $\mu$ ,  $0 \leq \mu \leq \frac{\pi}{4}$ , the angle at  $T$  measured clockwise in a horizontal plane from the base line  $TR$  to the vertical plane of the transmitter center beam; and by  $\alpha$ ,  $-\mu < \alpha < \pi - \mu$ , the angle at  $T$  measured counterclockwise in a horizontal plane from the base line  $TR$  to a line through  $T$  parallel to the satellite path (figure 3).

In the horizontal plane at some altitude  $x$ , the satellite will lie on the ellipse

$$\frac{(z+ae)^2}{\frac{a^2}{b^2}(b^2-x^2)} + \frac{y^2}{b^2-x^2} = 1, \quad (2.27)$$

where  $a$ ,  $b$  are the semi-major axis  $p_0$  and semi-minor axis respectively of the ellipsoid at closest approach. Since the satellite path is tangent to this ellipse at the location of the satellite, this point is to be determined from the expressions

$$y^2 = \frac{\frac{b^2}{a^2}(b^2-x^2)}{\frac{b^2}{a^2} + \tan^2 \alpha} \quad (2.28)$$

$$(z+ae)^2 = \frac{\frac{a^2}{b^2}(b^2-x^2) \tan^2 \alpha}{\frac{b^2}{a^2} + \tan^2 \alpha}. \quad (2.29)$$

The velocity components of the satellite are to be determined from the expressions

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y}^2 &= v^2 \sin^2 \alpha \\ \dot{z}^2 &= v^2 \cos^2 \alpha, \quad v^2 = \frac{K^2 M}{r+x}. \end{aligned} \quad (2.30)$$

Clearly, the problem is not yet solved, first because  $\alpha$  is not yet known, and, in addition, we must establish a criterion for determining the signs of  $y$ ,  $z+ae$ ,  $\dot{y}$  and  $\dot{z}$ .

From figure 3 it is clear that for

$$-\mu < \alpha \leq \frac{\pi}{2} - \mu, \quad (2.31)$$

$$\alpha \text{ is given by } \alpha_I = \theta - \mu. \quad (2.32)$$

For

$$\frac{\pi}{2} - \mu < \alpha < \pi - \mu, \quad (2.33)$$



$\alpha$  is given by

$$\alpha_{II} = \pi - (\theta + \mu) . \quad (2.34)$$

Due to the manner in which  $\theta$  is determined (2.25), we do not know which expression is the correct one for  $\alpha$ . We do know, however, that all possible paths are uniquely defined either by  $\alpha_I$  or by  $\alpha_{II}$  \*.

From (2.32) and (2.34),

$$\begin{aligned} \text{if } 0 < \theta \leq \mu & \quad \text{then } -\mu < \alpha_I \leq 0 \quad \text{and} \quad \pi - 2\mu \leq \alpha_{II} < \pi - \mu ; \\ \text{if } \mu < \theta < \frac{\pi}{2} - \mu & \quad \text{then } 0 < \alpha_I < \frac{\pi}{2} - 2\mu \quad \text{and} \quad \frac{\pi}{2} < \alpha_{II} < \pi - 2\mu ; \\ \text{if } \frac{\pi}{2} - \mu \leq \theta \leq \frac{\pi}{2} & \quad \text{then } \frac{\pi}{2} - 2\mu \leq \alpha_I \leq \frac{\pi}{2} - \mu \quad \text{and} \quad \frac{\pi}{2} - \mu \leq \alpha_{II} \leq \frac{\pi}{2} . \end{aligned}$$

Further, it is clear from figure 3 that

if  $-\mu < \alpha < 0$ , the satellite lies in the 1st quadrant ( $Q_1$ ) or 3rd quadrant ( $Q_3$ ) of the ellipse (2.27);

if  $0 < \alpha < \frac{\pi}{2}$ , the satellite lies in the 2nd or 4th quadrant ( $Q_2$  or  $Q_4$ );

if  $\frac{\pi}{2} < \alpha < \pi - \mu$ , the satellite lies in the 1st or 3rd quadrant ( $Q_1$  or  $Q_3$ ).

The relationships that exist between the computed values of  $\theta$ ,  $\alpha_I$ , and  $\alpha_{II}$  and the quadrants in which the satellite might lie are shown in figure 4. In general, a given value of  $\theta$  determines four possible points on the ellipse, consisting of two pairs of symmetric points. A unique velocity is associated with each pair. If all four points lie in the 1st and 3rd quadrants, then the direction of the satellite is clear, i.e.,  $\dot{y}/\dot{z} > 0$ . Further, the sign of  $\dot{y}$  is determined by observing, from the data, which beam the satellite crosses first. \*\* Hence the sign of  $\dot{z}$  is known. Similarly, if all four points lie in the 2nd and 4th quadrants,  $\dot{y}/\dot{z} < 0$ .

\*If the center beam is rotated above the base line, and if  $\mu$  is measured counterclockwise from the base line to the beam, the formulas will be different but the results will be the same.

\*\*The satellite of figure 2 is travelling in a southerly direction, hence  $\dot{y} < 0$ .

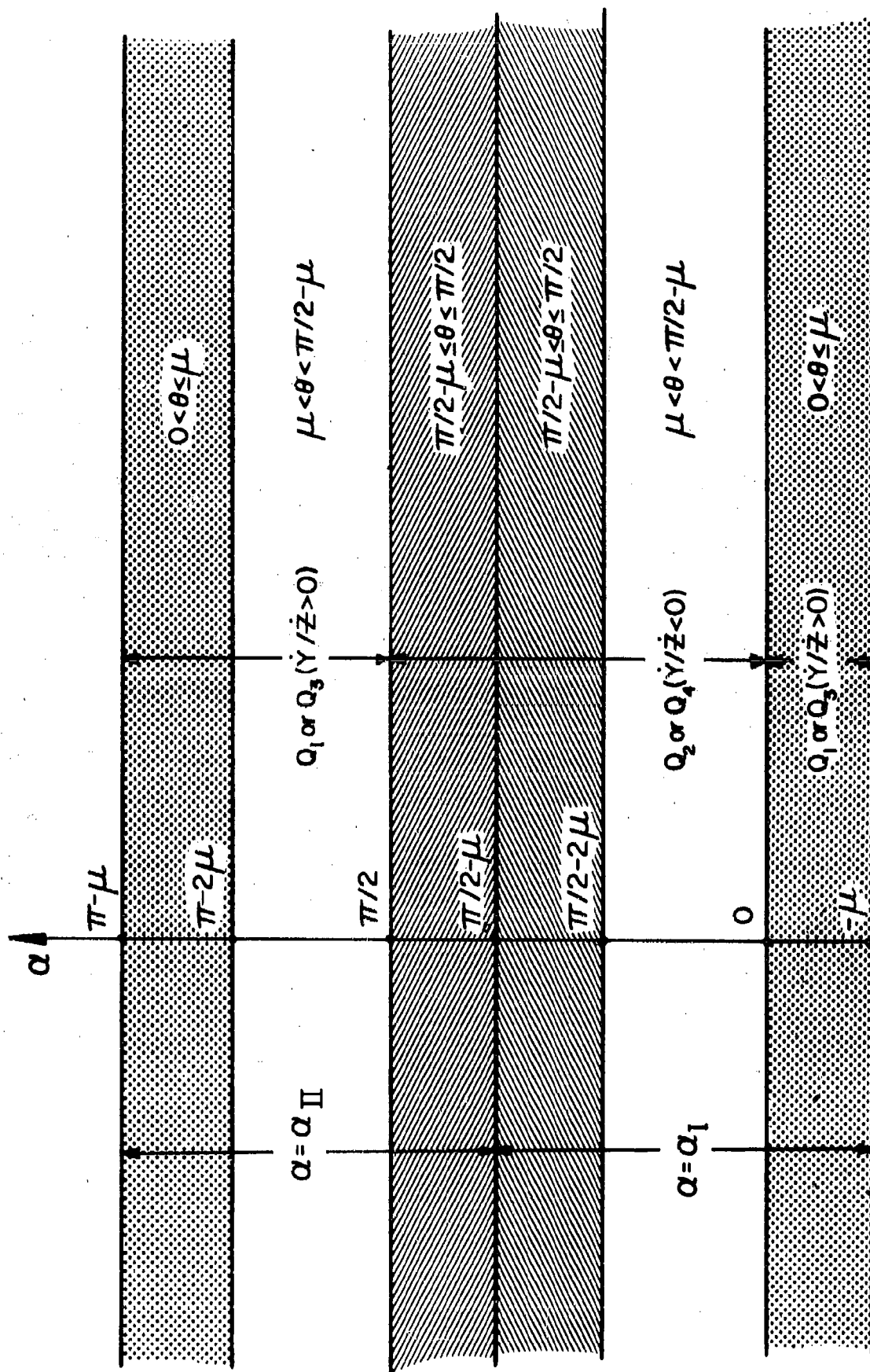


FIG.4-RELATIONS BETWEEN SATELLITE DIRECTION AND ITS LOCATION  
ON THE ELLIPSOID

Thus, the following conclusions have been established:

For a given  $\mu$ ,  $\Delta t$  and altitude  $x$ , and hence  $\theta$  (2.26), and denoting

$$B_1 = \frac{b}{a} \left[ \frac{b^2 - x^2}{\frac{b^2}{a^2} + \tan^2(\theta - \mu)} \right]^{\frac{1}{2}}; \quad B_2 = \frac{a}{b} \left[ \frac{b^2 - x^2}{\frac{b^2}{a^2} + \tan^2(\theta - \mu)} \right]^{\frac{1}{2}} \tan(\theta - \mu);$$

$$C_1 = \frac{b}{a} \left[ \frac{b^2 - x^2}{\frac{b^2}{a^2} + \tan^2(\theta + \mu)} \right]^{\frac{1}{2}}; \quad C_2 = -\frac{a}{b} \left[ \frac{b^2 - x^2}{\frac{b^2}{a^2} + \tan^2(\theta + \mu)} \right]^{\frac{1}{2}} \tan(\theta + \mu);$$

If  $0 < \theta \leq \mu$ , there are two possible points in 1st quadrant and two possible points in 3rd quadrant. Since  $\dot{y}/\dot{z} > 0$  for all these points, the general direction of motion of the satellite is known.

Points in 1st quad:  $y = B_1$ ,  $z = -ae + B_2$ ;  $\dot{y} = \mp v \sin(\theta - \mu)^*$ ,  $\dot{z} = \pm v \cos(\theta - \mu)$   
 $y = C_1$ ,  $z = -ae + C_2$ ;  $\dot{y} = \pm v \sin(\theta + \mu)$ ,  $\dot{z} = \pm v \cos(\theta + \mu)$

Points in 3rd quad:  $y = -B_1$ ,  $z = -ae - B_2$ ;  $\dot{y} = \mp v \sin(\theta - \mu)$ ,  $\dot{z} = \pm v \cos(\theta - \mu)$   
 $y = -C_1$ ,  $z = -ae - C_2$ ;  $\dot{y} = \pm v \sin(\theta + \mu)$ ,  $\dot{z} = \pm v \cos(\theta + \mu)$ ;

If  $\mu < \theta < \frac{\pi}{2} - \mu$ , there is one possible point in each of the four quadrants.  
 $(B_2 > 0, C_2 < 0)$  The direction of motion of the satellite is not known. \*\*

\*The sign of  $\dot{y}$  is to be determined from the data for each satellite.

\*\*Although the sign of  $\dot{y}$  is known from the data, the sign of  $\dot{z}$  depends upon whether the satellite is either in the 1st or 3rd quadrants, or in the 2nd or 4th quadrants. This is not known in this case.

Point in 1st quad:  $y = C_1, z = -ae + C_2; \dot{y} = \pm v \sin(\theta + \mu), \dot{z} = \pm v \cos(\theta + \mu)$   
 Point in 2nd quad:  $y = B_1, z = -ae + B_2; \dot{y} = \pm v \sin(\theta - \mu), \dot{z} = \mp v \cos(\theta - \mu)$   
 Point in 3rd quad:  $y = -C_1, z = -ae - C_2; \dot{y} = \pm v \sin(\theta + \mu), \dot{z} = \pm v \cos(\theta + \mu)$   
 Point in 4th quad:  $y = -B_1, z = -ae - B_2; \dot{y} = \pm v \sin(\theta - \mu), \dot{z} = \mp v \cos(\theta - \mu);$

If  $\frac{\pi}{2} - \mu \leq \theta \leq \frac{\pi}{2}$ , there are two possible points in the second quadrant and two possible points in the 4th quadrant. Since  $\dot{y}/\dot{z} < 0$  for all these points, the general direction of motion of the satellite is known.

Points in 2nd quad:  $y = B_1, z = -ae + B_2; \dot{y} = \pm v \sin(\theta - \mu), \dot{z} = \mp v \cos(\theta - \mu)$   
 $y = C_1, z = -ae + C_2; \dot{y} = \pm v \sin(\theta + \mu), \dot{z} = \pm v \cos(\theta + \mu)$

Points in 4th quad:  $y = -B_1, z = -ae - B_2; \dot{y} = \pm v \sin(\theta - \mu), \dot{z} = \mp v \cos(\theta - \mu)$   
 $y = -C_1, z = -ae - C_2; \dot{y} = \pm v \sin(\theta + \mu), \dot{z} = \pm v \cos(\theta + \mu).$

Clearly, the ambiguity of direction in the second case ( $\mu < \theta < \frac{\pi}{2} - \mu$ ) is removed if  $\mu = \frac{\pi}{4}$ , i. e., if the transmitter antenna is rotated  $45^\circ$  from the base line.

We still have the problem of selecting the correct altitude  $x$ , and the correct point  $(y, z)$  at this altitude, out of four possible values of  $(y, z)$ . We have seen that the possible values of  $x$  are already restricted by the upper bound of  $\sin \Theta$  (2.26). For a chosen  $x$ , the four possible values of  $(y, z)$  just defined are not all acceptable. The data tells us where the satellite is, at  $t_0$ , relative to the center beam. That is, the satellite is either in the beam or on one side of it, and proceeding toward it or away from it. This information, expressed analytically, imposes severe restrictions on the possible values of  $(y, z)$  that are to be accepted.

We first construct a new coordinate system  $x', y', z'$ , by rotating the  $y$  and  $z$  axes of figure 3 clockwise about the  $x$ -axis through the angle  $\mu$ . The  $x'y'$  plane is then vertical and normal to the center beam

(figure 5). Denote by  $t_1$  and  $t_2$  the times at which the satellite crosses the edges of the center beam. The distance travelled by the satellite along its path in the horizontal plane at altitude  $x = x'$  between the time of closest approach  $t_0$  and the time it crosses the vertical plane  $x'z'$  is given by

$$\Delta s' = v \left| \frac{t_1 + t_2}{2} - t_0 \right|.$$

Then, for the case where  $\dot{y}' > 0$  (satellite travelling north), the  $y'$  coordinate of the satellite at  $t_0$  is given by

$$y'_0 = v \sin \theta \left( t_0 - \frac{t_1 + t_2}{2} \right),$$

and for  $\dot{y}' < 0$  (satellite moving south), the  $y'$  coordinate at  $t_0$  is given by

$$y'_0 = v \sin \theta \left( \frac{t_1 + t_2}{2} - t_0 \right).$$

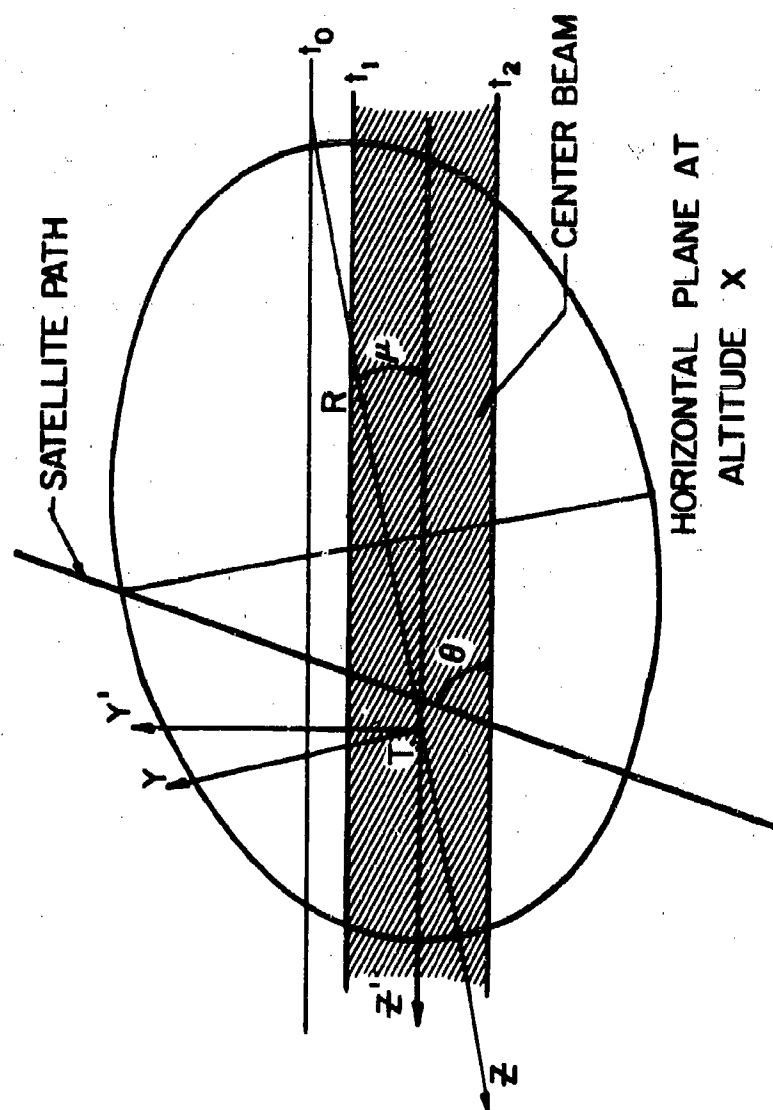
Since  $y' = y \cos \mu - z \sin \mu$ , the appropriate formula can be used as a criterion for selection of permissible points  $(y, z)$  at a given altitude  $x$ , from the four points previously computed. To allow for possible inaccuracy in the precise location of the satellite at times  $t_1$ ,  $t_2$ , and  $t_0$ , we can accept values of the  $y'$  coordinate that lie in the range  $y'_0 - \epsilon < y' < y'_0 + \epsilon$ ,  $\epsilon > 0$ . To select this range, define the angle  $\phi$  in the  $x'y'$  plane by the relation

$$\tan \phi = \frac{y'_0}{x'}.$$

Then, assigning the small angle  $\Delta \phi > 0$ , the  $x, y, z$  coordinates of the satellite must satisfy the inequality

$$\tan(\phi - \Delta \phi) < \frac{y'}{x'} = \frac{y \cos \mu - z \sin \mu}{x} < \tan(\phi + \Delta \phi). \quad (2.35)$$

Application to Doppler Field Data. - Using the methods just described, preliminary orbits were established for two revolutions of the Discoverer XI satellite. The data was obtained from the ARPA - BRL DOPLOC system,



VERTICAL PLANE AT T IN  
DIRECTION NORMAL TO BEAM

FIG.5—LOCATION OF SATELLITE RELATIVE TO CENTER BEAM

using the transmitter at Fort Sill, Oklahoma, and the receiver at Forrest City, Arkansas. The transmitter operated continuously; the receiver was turned on at one second intervals and operated until 1000 cycles of Doppler were counted. The recorded data consisted of the time duration of these 1000 cycle Doppler intervals, together with the starting time of each interval (figures 6A and 6B). The average frequency shift over each 1000 cycle Doppler interval was computed from the data by the formula

$$\overline{\Delta f_i} = \frac{1000}{\Delta t_i} - 7000,$$

where  $\Delta t_i$  is the duration of the Doppler interval, and the quantity 7000 represents an empirical constant of the particular transmitter-receiver system. This was converted to the range rate by

$$\dot{p}_i = \frac{1}{2} \lambda_r \overline{\Delta f_i}$$

and was assigned to be the value of  $\dot{p}$  at the midpoint of the associated Doppler interval  $\Delta t_i$ .

The preliminary orbits obtained are given in figure 7. For comparison, the final solutions obtained by BRL and NSSCC (Space Track) are also listed.\* BRL uses an analog procedure to determine a preliminary orbit from the DOPLOC data, while at Space Track all available data is compiled over a period of time to compute the orbit.

\*Space Track results were furnished to the author by BRL.

# DOPLOC DATA

DISCOVERER XI REV. #30 APRIL 17, 1960

First entry is the starting time of each 1000 cycle Doppler interval (Universal time).

Second entry is the duration, in seconds, of each 1000 cycle Doppler interval.

North Antenna		Center Antenna		South Antenna	
18:38:12.00000	.39951	18:39:19.00000	.14527	18:40:06.00000	.09331
13.	.39621	20.	.14259	07.	.09296
14.	.39216	21.	.14000	08.	.09262
15.	.38976	22.	.13751	09.	.09231
16.	.38638	23.	.13519	10.	.09206
17.	.38300	24.	.13297	11.	.09172
18.	.37963			12.	.09148
19.	.37615			13.	.09121
20.	.37263			14.	.09086
21.	.36913			15.	.09065
22.	.36558				
23.	.36210				
24.	.35851				
25.	.35485				
26.	.35125				
27.	.34749				
28.	.34371				
29.	.34003				
30.	.33760				

FIGURE 6 A



# DOPLOC DATA

DISCOVERER XI REV. #172 APRIL 26, 1960

First entry is the starting time of each 1000 cycle Doppler interval (Universal time).

Second entry is the duration, in seconds, of each 1000 cycle Doppler interval.

	North Antenna	Center Antenna	South Antenna
16:53:08.00000	.31132	16:53:56.00000 .14509	16:54:29.00000 .09470
09.	.30822	57. .14168	30. .09428
10.	.30512	58. .13839	31. .09389
11.	.30199	59. .13523	32. .09350
12.	.29885	60. .13224	33. .09314
13.	.29574		34. .09279
14.	.29264		35. .09246
15.	.28951		36. .09213
16.	.28636		37. .09182
17.	.28317		38. .09156
18.	.28003		
19.	.27685		
20.	.27362		
21.	.27044		
22.	.26718		
23.	.26399		
24.	.26075		
25.	.25751		
26.	.25422		
27.	.25107		

FIGURE 6 B

DISCOVERER XI    REV. #30    APRIL 17, 1960

	a	e	M(t) t = 18:39:21.5	i	$\Omega$	$\omega$
Space Sciences Laboratory (Preliminary Orbit)	4090 mi.	.0106	342.48°	76.41°	217.7°	144.18°
BRL (Final Solution)	4186 mi.	.0295	345.23°	80.15°	215.93°	154.97°
Space Track (Final Solution)	4188 mi.	.0291	- -	80.01°	215.84°	146.28°

REV. #172    APRIL 26, 1960

	a	e	M(t) t = 16:53:58	i	$\Omega$	$\omega$
Space Sciences Laboratory (Preliminary Orbit)	4045 mi.	.0147	41.21°	74.5°	204.8°	59.31°
BRL (Final Solution)	4093 mi.	.0189	43.01°	80.44°	203.18°	96.08°
Space Track (Final Solution)	4090 mi.	.0111	- -	80.10°	202.84°	114.23°

FIGURE 7

### 3. ESTABLISHING A PRELIMINARY ORBIT FOR ACTIVE SATELLITES

When the transmitter is aboard the satellite,  $\bar{\rho}_T \equiv 0$  and the Doppler equation for a single receiver is \*

$$\begin{aligned} \dot{\bar{\rho}}_R &= \lambda_T \Delta f, \\ \Delta f &= f_T - f_R, \quad \lambda_T = c/f_T \end{aligned} \quad (3.1)$$

Integration of this equation discloses that at any given time  $t$  the satellite lies on the surface of a sphere of radius  $\rho_R(t)$ , with center at the receiver.

Eliminating the subscript R for convenience, the vector form of (3.1) is

$$\frac{\bar{\rho} \cdot \dot{\bar{\rho}}}{\rho} = \lambda_T \Delta f,$$

and again neglecting the earth's rotation,

$$\bar{v} \cdot \frac{\bar{\rho}}{\rho} = \lambda_T \Delta f, \quad (3.2)$$

where  $\bar{v}$  is the satellite velocity.

Now define the time of closest approach as the time at which  $\rho$  is minimum. Then the following conclusions have been established for an active satellite:

At the time of closest approach  $t_0$  the satellite lies on the surface of the sphere of radius  $\rho_0$ , with center at the receiver R; further, the satellite trajectory is tangent to this sphere.

If we now assume that  $\bar{v} = \dot{\bar{\rho}}$  is constant in the neighborhood of closest approach  $t_0$ , then

$$\bar{\rho}(t) = \bar{\rho}_0 + \bar{v}_c(t - t_0),$$

\* See appendix.

and, in analogy to (2.13),

$$\dot{\rho}(t) = \frac{v_o^2(t-t_o)}{[\rho_o^2 + v_o^2(t-t_o)^2]^{3/2}} \quad (3.3)$$

Thus, in the case of the active satellite, the parameters at closest approach,  $\rho_o$ ,  $v_o$ ,  $t_o$ , are related to the Doppler data by (3.1) and (3.3). These parameters are computed in the same way as were the passive satellite parameters  $\rho_o$ ,  $|\dot{\vec{p}}_o|$ ,  $t_o$ .

The procedure for determining the satellite position and velocity components at closest approach is quite analogous to that for the passive satellite, except that now the satellite speed at closest approach is known ( $v = v_o$ ).

If, corresponding to the base line TR of the ellipse of figure 3, we define the parameters  $\mu$  and  $\alpha$  for the circle of figure 8 with respect to the latitude of the receiver R, then, as before,

$$\alpha = \alpha_I = \theta - \mu, \quad -\mu < \alpha \leq \frac{\pi}{2} - \mu;$$

$$\alpha = \alpha_{II} = \pi - \theta + \mu, \quad \frac{\pi}{2} - \mu < \alpha < \pi - \mu.$$

We will place the origin of the topocentric coordinate system at R, with x-axis vertical, y-axis due North, and z-axis due West. Then the first quadrant of the circle is the north-east sector, the second quadrant the north-west sector, and so forth (figure 8). Thus, the relationships that exist between the computed values of  $\theta$ ,  $\alpha_I$ , and  $\alpha_{II}$ , and the quadrants in which the satellite might lie are again given by figure 4.

The active satellite, at altitude  $x$ , will lie on the circle

$$y^2 + z^2 = \rho_o^2 - x^2.$$

Since the satellite path is tangent to this circle, the coordinates

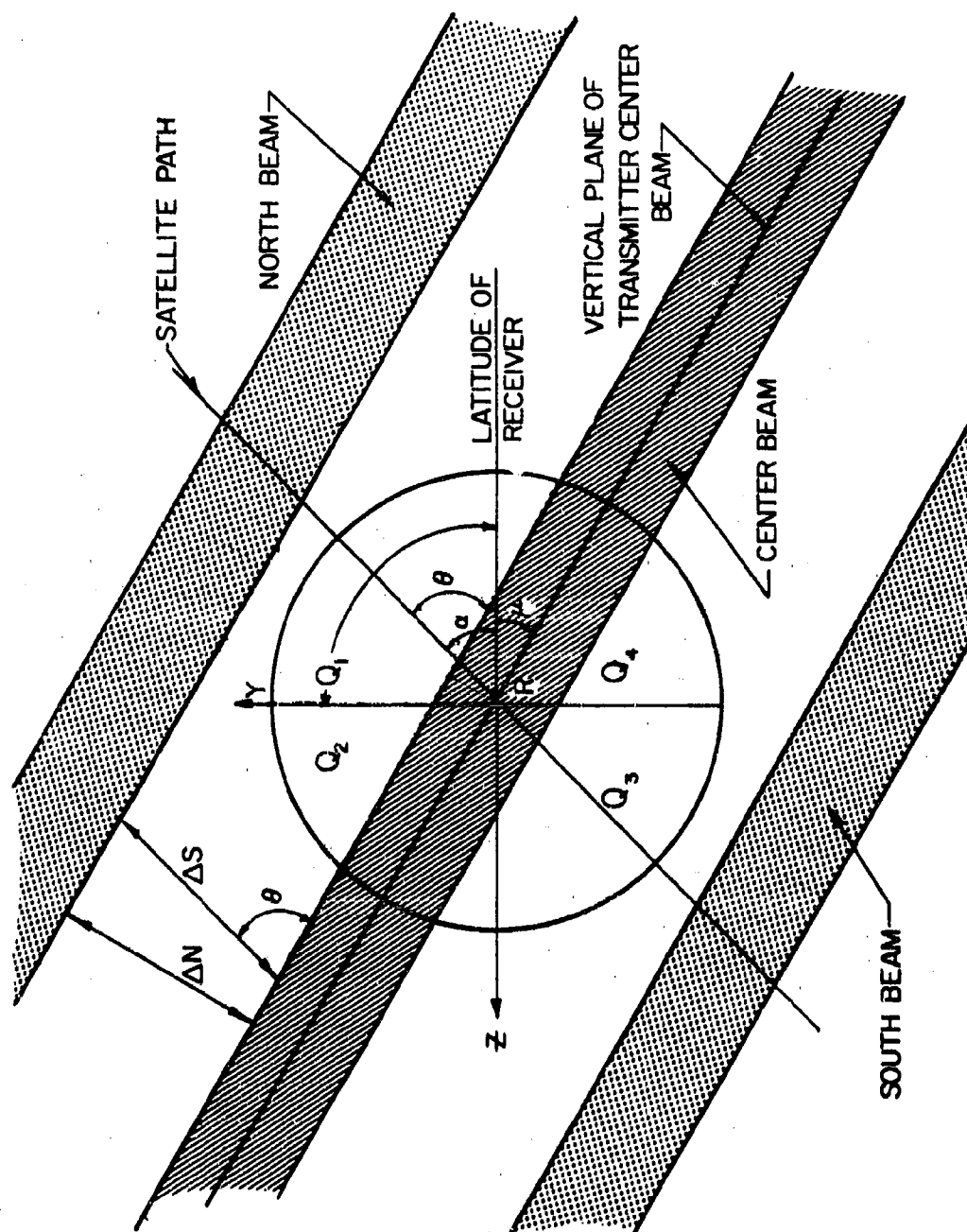


FIG. 8—ORIENTATION OF SPHERE AND ANTENNA BEAMS

of the satellite are determined from the expressions

$$y^2 = (\rho_c^2 - x^2) \cos^2 \alpha$$

$$z^2 = (\rho_c^2 - x^2) \sin^2 \alpha.$$

As before, the velocity components are obtained from

$$\dot{x} = 0, \quad \dot{y}^2 = v^2 \sin^2 \alpha, \quad \dot{z}^2 = v^2 \cos^2 \alpha.$$

The following conclusions are then established for the active satellite:

Given  $v = v_0$ ,  $\Delta t$ ,  $\mu$ , altitude  $x$  and  $\theta = \arcsin \frac{Ax}{v\Delta t}$ ,

$$A = \cot 28^\circ - \tan 4^\circ,$$

and denoting

$$D_1 = (\rho_c^2 - x^2)^{1/2} \cos(\theta - \mu), \quad D_2 = (\rho_c^2 - x^2)^{1/2} \sin(\theta - \mu),$$

$$F_1 = (\rho_c^2 - x^2)^{1/2} \cos(\theta + \mu), \quad F_2 = (\rho_c^2 - x^2)^{1/2} \sin(\theta + \mu):$$

If  $0 < \theta \leq \mu$ , there are two possible points in 1st quadrant and two possible points in 3rd quadrant. Since  $\dot{y}/\dot{z} > 0$  for all these points, the general direction of motion of the satellite is known.

Points in 1st quad:  $y = D_1, z = D_2; \dot{y} = \mp v \sin(\theta - \mu)^*$ ,  $\dot{z} = \pm v \cos(\theta - \mu)$   
 $y = F_1, z = -F_2; \dot{y} = \pm v \sin(\theta + \mu)$ ,  $\dot{z} = \mp v \cos(\theta + \mu)$

Points in 3rd quad:  $y = -D_1, z = -D_2; \dot{y} = \mp v \sin(\theta - \mu)$ ,  $\dot{z} = \pm v \cos(\theta - \mu)$   
 $y = -F_1, z = F_2; \dot{y} = \pm v \sin(\theta + \mu)$ ,  $\dot{z} = \mp v \cos(\theta + \mu);$

\* The sign of  $\dot{y}$  is to be determined from the data for each satellite.

If  $\mu < \theta < \frac{\pi}{2} - \mu$ , there is one possible point in each of the four quadrants.

$$(D_1 > 0, D_2 > 0, \\ F_1 > 0, F_2 > 0).$$

The direction of motion of the satellite is not known.\*

Point in 1st quad:  $y = F_1, z = -F_2; \dot{y} = \pm v \sin(\theta + \mu), \dot{z} = \pm v \cos(\theta + \mu)$

Point in 2nd quad:  $y = D_1, z = D_2; \dot{y} = \pm v \sin(\theta - \mu), \dot{z} = \mp v \cos(\theta - \mu)$

Point in 3rd quad:  $y = -F_1, z = F_2; \dot{y} = \pm v \sin(\theta + \mu), \dot{z} = \pm v \cos(\theta + \mu)$

Point in 4th quad:  $y = -D_1, z = -D_2; \dot{y} = \pm v \sin(\theta - \mu), \dot{z} = \mp v \cos(\theta - \mu);$

If  $\frac{\pi}{2} - \mu \leq \theta \leq \frac{\pi}{2}$ , there are two possible points in the second quadrant

$$(D_1 > 0, D_2 > 0, \\ F_1 < 0, F_2 > 0)$$

and two possible points in the 4th quadrant. Since  $\dot{y}/\dot{z} < 0$  for all these points, the general direction of motion of the satellite is known.

Points in 2nd quad:  $y = D_1, z = D_2; \dot{y} = \pm v \sin(\theta - \mu), \dot{z} = \mp v \cos(\theta - \mu)$

$y = -F_1, z = F_2; \dot{y} = \pm v \sin(\theta + \mu), \dot{z} = \pm v \cos(\theta + \mu)$

Points in 4th quad:  $y = -D_1, z = -D_2; \dot{y} = \pm v \sin(\theta - \mu), \dot{z} = \mp v \cos(\theta - \mu)$

$y = F_1, z = -F_2; \dot{y} = \pm v \sin(\theta + \mu), \dot{z} = \pm v \cos(\theta + \mu).$

The ambiguity of direction in the second case ( $\mu < \theta < \frac{\pi}{2} - \mu$ ) is removed if  $\mu = \frac{\pi}{4}$ , i. e., if the transmitter antenna is rotated  $45^\circ$  from the latitude of the receiver.

Finally, the location of the satellite relative to the center beam can be used as a criterion for selecting permissible points ( $y, z$ ) at a given altitude  $x$ , from the four points just computed. As in the case of the passive satellite, this criterion is given by inequality (2.35).

\* Although the sign of  $\dot{y}$  is known from the data, the sign of  $\dot{z}$  depends upon whether the satellite is either in the 1st or 3rd quadrants, or in the 2nd or 4th quadrants. This is not known in this case.

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## 5. APPENDIX

### Derivation of the Doppler Equation for a Passive Satellite.

We will assume that radio waves travel with constant velocity of light through all disturbances, and that the motion of the satellite is slow compared to the speed of light.

We consider a stationary transmitter  $T$  on the surface of the earth at some positive distance from a stationary receiver  $R$  on the surface of the earth. A signal emitted at  $T$  at time  $t$ , travelling at the speed of light  $c$ , arrives at the moving satellite  $S$  at time

$$\tau = t + \frac{\rho_T(\tau)}{c},$$

where  $\rho_T$  is the magnitude of the position vector  $\vec{\rho}_T$  of the satellite  $S$  relative to the transmitter  $T$ . Denote by  $p_T$  an active transmission time interval and by  $p_s$  the corresponding time interval during which the satellite is illuminated. Then

$$p_s = p_T + \frac{\rho_T(\tau + p_s) - \rho_T(\tau)}{c}.$$

If the relative range  $\rho_T$  of the satellite is assumed to be a linear function of time during this interval, we obtain, dividing by  $p_s$ ,

$$1 = \frac{p_T}{p_s} + \frac{\dot{\rho}_T}{c},$$

which yields

$$\frac{f_s}{f_T} = 1 - \frac{\dot{\rho}_T}{c}, \quad (5.1)$$

where  $f_s$  and  $f_T$  are the reflected (satellite) and transmitted frequencies, respectively.

In a similar manner, the relationship between the satellite illumination interval and the interval during which the reflected signal is received at  $R$  is expressed in terms of the satellite frequency  $f_s$  and the receiver frequency  $f_R$  by the equation

$$\frac{f_R}{f_s} = 1 - \frac{\dot{\rho}_R}{c}, \quad (5.2)$$

where higher order terms in  $\dot{e}_R/c$  have been neglected. From (5.1) and (5.2) we obtain the following expression for the Doppler frequency shift of a passive satellite:

$$f_T - f_R = \frac{f_T}{c} (\dot{e}_T + \dot{e}_R) - \frac{f_T}{c^2} \dot{e}_T \dot{e}_R ,$$

and, again neglecting second order terms in  $\dot{e}/c$ , this simplifies to the Doppler equation for a passive satellite,

$$\dot{e}_T + \dot{e}_R = \frac{c}{f_T} (f_T - f_R) . \quad (5.3)$$

If the transmitter is aboard the satellite,  $\bar{e}_T \equiv 0$  and equation (5.3) reduces to the Doppler equation for an active satellite,

$$\dot{e}_R = \frac{c}{f_T} (f_T - f_R)$$

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